PREPUBLICACIONES DEL DEPARTAMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2013-01

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Enero-2013

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THIN DOMAINS WITH DOUBLY OSCILLATORY BOUNDARY †

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Abstract. We consider a 2-dimensional thin domain with order of thickness ϵ which presents oscillations of amplitude also ϵ on both boundaries , top and bottom, but the period of the oscillations are of different order at the top and at the bottom. We study the behavior of the Laplace operator with Neumann boundary condition and obtain its asymptotic homogenized limit as $\epsilon \to 0$. We are interested in understanding how this different oscillatory behavior at the boundary, influences the limit problem.

1. Introduction

In this paper, we analyze the behavior of the solutions of the Laplace equation with homogeneous Neumann boundary conditions

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f^\epsilon \quad \text{in } R^\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 \quad \text{on } \partial R^\epsilon \end{cases} \tag{1.1}$$

with $f^\epsilon \in L^2(R^\epsilon)$ and N^ϵ is the unit outward normal to ∂R^ϵ. The domain R^ϵ is a two dimensional thin domain which presents a highly oscillatory behavior at the boundary and it is given as the region between two oscillatory functions, that is,

$$R^\epsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon\, h(x_1/\epsilon^\alpha) < x_2 < \epsilon\, g(x_1/\epsilon)\Big\}, \quad \text{with } \alpha > 1. \tag{1.2}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are C^1 periodic functions with period L_1 and L_2 respectively (see Figure 1). Moreover, there exist constants $h_0 \geq 0$ and $h_1, g_0, g_1 > 0$ such that $0 \leq h_0 \leq h(\cdot) \leq h_1$, and $0 < g_0 \leq g(\cdot) \leq g_1$.

Observe that both the amplitude and period of the oscillations at the upper boundary, given by $\epsilon g(x/\epsilon)$ are of the same order as the thickness of the domain. But, for the lower boundary, which is given by $\epsilon h(x/\epsilon^\alpha)$, the amplitude is of the same order ϵ, while the period is of the order of ϵ^α, which means that we have much more oscillations at the bottom than at the top boundary. order of the lower oscillations is large than the order of the amplitude and height of the thin domain R^ϵ with respect to the small parameter ϵ.

The existence and uniqueness of solutions for problem (1.1) for each $\epsilon > 0$, is guaranteed by Lax-Milgram Theorem. We will analyze the asymptotic behavior of the solutions as $\epsilon \to 0$.

Since the domain is thin, $R^\epsilon \subset (0,1) \times (-\epsilon h(\cdot), \epsilon g(\cdot))$, approaching the interval $(0,1)$, it is reasonable to expect that the family of solutions will converge to a function of just one variable and that this function will satisfy certain elliptic equation in one dimension with some boundary conditions. As a matter of fact, if the function $h_\epsilon(\cdot)$ is independent of ϵ, say $h_\epsilon(\cdot) \equiv 0$, the limit equation is given by

$$\begin{cases} -q_0 w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{1.3}$$

where $q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2$, and X is a convenient auxiliary harmonic function defined in the representative basic cell $Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < l, \quad 0 < y_2 < G(y_1)\}$.

The purely periodic case can be addressed by somehow standard techniques in homogenization theory, as accomplished in [1, 2]. See [5, 9] for general references in homogenization and [7] for reticulated structures. Observe that in this case the extension operators are very important for the convergence proof.

Key words and phrases. thin domains, oscillatory boundary, homogenization.

† Partially supported by grant MTM2012-31298, MINECO, Spain and Grupo de Investigación CADEDIF, UCM.

Figure 1. Thin domain R^ϵ

Moreover, if we assume that $g_\epsilon(\cdot)$ is independent of ϵ, say $g_\epsilon(\cdot) = g(\cdot)$, and $h_0 = \min_{x\in\mathbb{R}}\{h(x)\}$ then the variational formulation of the limit problem is:

$$\int_0^1 \Big\{ \Big(g(x) + h_0\Big)\, w_x(x)\,\varphi_x(x) + p(x)\,\omega(x)\,\varphi(x)\Big\}dx = \int_0^1 \hat{f}(x)\,\varphi\,dx, \quad \forall\varphi\in H^1(0,1) \tag{1.4}$$

where $p(x) = g(x) + \frac{1}{L_2}\int_0^{L_2} h(s)\,ds$, for all $x\in(0,1)$, and the function $\hat{f}^\epsilon(x) = \int_{-h_\epsilon(x)}^{g(x)} f(x,y)\,dy$ satisfies that $\hat{f}^\epsilon \rightharpoonup \hat{f}$, w-$L^2(0,1)$. We refer to [3] for details. In this work, we want to analyze the case where the thin domain is a region between two functions with different order of the oscillations.

Our case is a combination of these two cases since both g_ϵ and h_ϵ are present. And we want to understand the effect of both terms at the same time in the limit equation. Notice that the techniques used to solve each case separately are different so we will need to combine both techniques to get the limit problem in our case. The main difference of the present work in relation to previous existing work in the literature, see for instance [6, 8, 4] and references therein, is that we allow two different order of oscillations in the boundary of the thin domain.

In Section 2 we state the notation and the problem that we will study. Furthermore, we are going to construct an extension operator that will be very important in the proof of the convergence result. Finally, we state the main convergence result.

In Section 3 we rigorously prove the convergence result. In order to do so, we combine two different techniques: we use an extension operator in the upper boundary and we define suitable rectangles in the lower boundary to apply the estimates that we obtained in Lemma 3.1 .

2. Notation and statement of main result

To study the convergence of the solutions of (1.1) we first perform the change of variables $(x,y)\to(x,\epsilon y)$, which transforms the domain R^ϵ into the domain Ω^ϵ

$$\Omega^\epsilon = \Big\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\in(0,1),\ -h(x_1/\epsilon^\alpha) < x_2 < g(x_1/\epsilon)\Big\}. \tag{2.1}$$

Under this transformation, we obtain the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial x_1}\nu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}\nu_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon, \end{cases} \tag{2.2}$$

where $f^\epsilon\in L^2(\Omega^\epsilon)$ satisfies $\|f^\epsilon\|_{L^2(\Omega^\epsilon)}\le C$, for some $C>0$ independent of ϵ, and $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the outward unit normal to $\partial\Omega^\epsilon$. Observe that Ω^ϵ is not a thin domain anymore but there appears a factor $1/\epsilon^2$ in front of the derivative in the x_2. Moreover, the domain has very wild oscillatory behavior at the top and bottom boundary.

For the analysis we will construct an extension operator for functions defined in the set Ω^ϵ, but which will extend the function only over the upper part of the boundary. Hence, let us consider the following open set:

$$\widetilde{\Omega}^\epsilon = \Big\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\in(0,1),\ -h(x_1/\epsilon^\alpha) < x_2 < g_1\Big\}. \tag{2.3}$$

Lemma 2.1. *With the notation above, there exists an extension operator*

$$P_\epsilon \in \mathcal{L}(L^p(\Omega^\epsilon),\, L^p(\widetilde{\Omega}^\epsilon)) \cap \mathcal{L}(W^{1,p}(\Omega^\epsilon),\, W^{1,p}(\widetilde{\Omega}^\epsilon))$$

such that for any $\varphi \in W^{1,p}(\Omega^\epsilon)$,

$$||P_\epsilon \varphi||_{L^p(\widetilde{\Omega}^\epsilon)} \le K||\varphi||_{L^p(\Omega^\epsilon)}, \ \left\|\frac{\partial P_\epsilon \varphi}{\partial x_2}\right\|_{L^p(\widetilde{\Omega}^\epsilon)} \le K \left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^p(\Omega^\epsilon)} \tag{2.4}$$

$$\text{and } \left\|\frac{\partial P_\epsilon \varphi}{\partial x_1}\right\|_{L^p(\widetilde{\Omega}^\epsilon)} \le K \left\{ \left\|\frac{\partial \varphi}{\partial x_1}\right\|_{L^p(\Omega^\epsilon)} + \eta(\epsilon) \left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^p(\Omega^\epsilon)} \right\} \tag{2.5}$$

$$\tag{2.6}$$

where $1 \le p \le \infty$, K *a constant independent of* ϵ *and* $\eta(\epsilon) = \sup_{x \in I}\{|g'_\epsilon(x)|\}$.

Proof. The extension operator is constructed with a reflection procedure over the upper boundary, as in [1]. □

Now, we state the convergence result:

Theorem 2.2. *Assume that* $f^\epsilon \in L^2(\Omega^\epsilon)$ *satisfies* $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$ *with* C *independent of the parameter* ϵ *and that there exists* $\hat{f} \in L^2(0,1)$ *such that* $\hat{f}^\epsilon \rightharpoonup \hat{f}$, $w - L^2(0,1)$, *where* $\hat{f}^\epsilon(x_1) \equiv \int_{-h(x_1/\epsilon^\alpha)}^{g_1} \widetilde{f}^\epsilon(x_1,x_2)\,dx_2$. *Let* u^ϵ *be the unique solution of (2.2). Then, there exists* $u_0 \in H^1(0,1)$ *such that if* P_ϵ *is the extension operator constructed in Lemma 2.1, we have* $\|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \to 0$ *and* u_0 *is the unique weak solution of the Neumann problem*

$$\int_0^1 \Big\{\hat{q}\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + (\frac{|Y^*|}{L_1} + p)\,u_0\,\varphi\Big\}dx_1 = \int_0^1 \hat{f}\,\varphi\,dx_1, \quad \forall \varphi \in H^1(0,1). \tag{2.7}$$

where Y^* *is the basic cell*

$$Y^* = \{(y_1,y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L_1 \ \textit{and} \ -h_0 < y_2 < g(y_1)\}.$$

The homogenized constant coefficients are defined by

$$\hat{q} \equiv \int_{-h_0}^{g_1} q(s)\,ds = \frac{1}{L_1}\int_{Y^*}\Big\{1 - \frac{\partial X}{\partial y_1}(y_1,y_2)\Big\}dy_1dy_2, \quad p = \frac{1}{L_2}\int_0^{L_2} h(s)ds - h_0, \tag{2.8}$$

where X *is the unique solution (up to constants) which is* L_1*-periodic in the first variable, of the problem:*

$$\begin{cases} -\Delta X = 0 \ \textit{in} \ Y^* \\ \dfrac{\partial X}{\partial N} = 0 \ \textit{on} \ B_2 \\ \dfrac{\partial X}{\partial N} = -\dfrac{g'(y_1)}{\sqrt{1+g'(y_1)^2}} \ \textit{on} \ B_1 \end{cases} \tag{2.9}$$

B_0 *is the lateral part of the boundary,* B_1 *is the upper boundary and* B_2 *is the lower boundary of* ∂Y^*.

Remark 2.3. *If the non homogeneous term* $f^\epsilon(x_1,x_2)$ *is a fixed function depending only on the first variable, that is,* $f^\epsilon(x_1,x_2) = f(x_1)$, *it is easy to see that* $\hat{f}(x_1) = (\frac{|Y^*|}{L_1} + p)f(x_1)$ *and therefore, (2.7) is the variational version of*

$$\begin{cases} -\dfrac{\hat{q}}{\frac{|Y^*|}{L_1} + p} w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{2.10}$$

Notice that in case $h(\cdot) \equiv 0$, *then* $p = 0$ *and* $\frac{\hat{q}}{|Y^*|/L_1} = \frac{1}{|Y^*|}\int_{Y^*}(1 - \frac{\partial X}{\partial y_1}) = q_0$ *and we recover (1.3).*

3. Proof of the main result

The variational formulation of (2.2) is: find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2, \quad \forall \varphi \in H^1(\Omega^\epsilon). \tag{3.1}$$

Taking $\varphi = u^\epsilon$ in expression (3.1) and using that $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$, we easily obtain the a priori bounds

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \le C. \tag{3.2}$$

If we denote by $\widetilde{\ }$ the standard extension by zero and by χ^ϵ the characteristic function of Ω^ϵ, we may write (3.1) as

$$\int_{\Omega_0} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} \Big\} + \int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\frac{\partial \varphi}{\partial x_2} \Big\} + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \varphi = \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon f^\epsilon \varphi \ \forall \varphi \in H^1(\Omega^\epsilon), \tag{3.3}$$

where we divide the domain $\widetilde{\Omega}^\epsilon$ in two parts: one of them, $\widetilde{\Omega}^\epsilon_-$, carries all the oscillations and the other Ω_0 is a fixed domain, that is,

$$\begin{aligned} \widetilde{\Omega}^\epsilon_- &= \{(x_1,x_2) \in \mathbb{R}^2 |\, x_1 \in (0,1),\, -h(x_1/\epsilon^\alpha) < x_2 < -h_0\} \\ \Omega_0 &= \{(x_1,x_2) \in \mathbb{R}^2 |\, x_1 \in (0,1),\, -h_0 < x_2 < g_1\}. \end{aligned} \tag{3.4}$$

Before we start with the proof of the main result, let us state some relevant estimates on the solutions of certain elliptics problems, posed in rectangles of the type

$$Q_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha,\, 0 < y < 1\}, \text{ with } \alpha > 1. \tag{3.5}$$

As a matter of fact, for $u_0(\cdot) \in H^1(-\epsilon^\alpha, \epsilon^\alpha)$, we define the function $u^\epsilon(x,y)$ as the unique solution of

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial x^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial y^2} = 0 & \text{in } Q_\epsilon, \\ u(x,0) = u_0(x), & \text{on } \Gamma_\epsilon, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial Q_\epsilon \setminus \Gamma_\epsilon \end{cases} \tag{3.6}$$

where ν is the outward unit normal to ∂Q_ϵ and $\Gamma_\epsilon = \{(x,0) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha\}$.

We have the following,

Lemma 3.1. *With the notation from above, if we denote by $\bar{u}_0$ the average of u_0 in Γ_ϵ, that is $\bar{u}_0 = \frac{1}{2\epsilon^\alpha}\int_{-\epsilon^\alpha}^{\epsilon^\alpha} u_0(x)\,dx$ then there exists a constant C, independent of ϵ and u_0, such that*

$$\int_0^1 \int_{-\epsilon^\alpha}^{\epsilon^\alpha} |u^\epsilon(x,y) - \bar{u}_0|^2\, dxdy \le C\epsilon^{\alpha-1}\|u_0\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)} \tag{3.7}$$

and

$$\Big\|\frac{\partial u^\epsilon}{\partial x}\Big\|^2_{L^2(Q_\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial u^\epsilon}{\partial y}\Big\|^2_{L^2(Q_\epsilon)} \le C\epsilon^{\alpha-1}\Big\|\frac{\partial u_0}{\partial x}\Big\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}. \tag{3.8}$$

Proof. See [3] for details. □

Proof of Theorem 2.2. The idea is to pass to the limit in (3.3) constructing appropriate test functions. First, we study the limit of the different functions that form the integrands of (3.3).

(a). Limit in the extended functions. Using the a priori estimate (3.2) and the results from Lemma 2.1 we obtain that $P_\epsilon u^\epsilon|_{\Omega_0} \in H^1(\Omega_0)$ and we can extract a subsequence of $\{P_\epsilon u^\epsilon|_{\Omega_0}\} \subset H^1(\Omega_0)$, denoted again by $P_\epsilon u^\epsilon$, such that

$$\begin{aligned} &P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega_0) \\ &\frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega_0) \end{aligned} \tag{3.9}$$

as $\epsilon \to 0$ for some $u_0 \in H^1(\Omega_0)$.

A consequence of the limits (3.9) is that $u_0(x_1, x_2)$ does not depend on the variable x_2. Moreover, we have that the restriction of $P_\epsilon u^\epsilon$ to the coordinate axis x_1 converges to u_0. That is, $P_\epsilon u^\epsilon|_\Gamma \to u_0 \quad s - H^s(\Gamma)$ for all $s \in [0, 1/2)$ where $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 \,|\, x_1 \in (0,1)\}$. Consequently, we obtain $\|P_\epsilon u^\epsilon - u_0\|_{L^2(\Gamma)} \to 0$ as $\epsilon \to 0$. In view of the above limit, one has the L^2-convergence of $P_\epsilon u^\epsilon$ to u_0, that is

$$\|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{3.10}$$

In fact, on the one hand we have

$$\begin{aligned} \|P_\epsilon u^\epsilon(x_1, 0) - u_0(x_1)\|^2_{L^2(\widetilde{\Omega}^\epsilon)} &= \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} |P_\epsilon u^\epsilon(x_1, 0) - u_0(x_1)|^2 \, dx_2 dx_1 \\ &\le C(g,h) \, \|P_\epsilon u^\epsilon - u_0\|_{L^2(\Gamma)} \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

On the other hand,

$$\|P_\epsilon u^\epsilon(x_1, x_2) - P_\epsilon u^\epsilon(x_1, 0)\|^2_{L^2(\widetilde{\Omega}^\epsilon)} = \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} |P_\epsilon u^\epsilon(x_1, x_2) - P_\epsilon u^\epsilon(x_1, 0)|^2 \, dx_1 dx_2$$

$$\le \int_0^1 \int_{-h(x_1/\epsilon^\alpha)}^{g_1} \left(\int_0^{x_2} \left| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2}(x_1, s) \right|^2 ds \right) |x_2| \, dx_2 dx_1 \le C(h,g) \left\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon)} \le \epsilon \, \hat{C}(h,g) \to 0 \text{ as } \epsilon \to 0.$$

Finally

$$\begin{aligned} \|P_\epsilon u^\epsilon - u_0\|_{L^2(\widetilde{\Omega}^\epsilon)} \quad &\le \quad \|P_\epsilon u^\epsilon(x_1, x_2) - P_\epsilon u^\epsilon(x_1, 0)\|_{L^2(\widetilde{\Omega}^\epsilon)} + \|P_\epsilon u^\epsilon(x_1, 0) - u_0(x_1)\|_{L^2(\widetilde{\Omega}^\epsilon)} \to 0, \\ &\text{as } \epsilon \to 0. \end{aligned}$$

(b). Limit in the tilde functions.

From the a priori estimates (3.2) we know that there exists a function $\xi^* \in L^2(\Omega_0)$, such that, up to subsequences

$$\widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \rightharpoonup \xi^* \; w - L^2(\Omega_0) \;\text{ and }\; \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \to 0 \; s - L^2(\Omega_0); \quad \text{as } \epsilon \to 0. \tag{3.11}$$

(c). Limit of χ^ϵ.

Let χ be the characteristic function of the representative cell Y^*. We extend χ periodically on the variable $y_1 \in \mathbb{R}$ and denote this extension again by χ. Clearly, by construction, $\chi^\epsilon(x_1, x_2) = \chi(x_1/\epsilon, x_2)$, for $(x_1, x_2) \in \Omega^\epsilon_+$.

Consequently, by the Average Theorem and the Lebesgue's Dominated Convergence Theorem we obtain

$$\chi^\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \theta \quad w^* - L^\infty(\Omega_0), \quad \text{where } \theta(x_2) := \frac{1}{L_1} \int_0^{L_1} \chi(s, x_2) ds \quad \forall x_2 \in (-h_0, g_1). \tag{3.12}$$

(d) Test functions.

In order to construct appropriate test functions that will allow us to pass the limit in the variational formulation (3.3), we are going to need to define a partition of the unit interval $[0,1]$ which is related to the function h_ϵ and which will allow us to analyze in detail the effect of the oscillations at the bottom in the limit equation. Hence, denote by N_ϵ the largest integer such that $N_\epsilon L_2 \epsilon^\alpha < 1$, where L_2 is the period of the function h. Observe that $N_\epsilon \sim L_2^{-1} \epsilon^{-\alpha}$. Let

$$h_{n,\epsilon} = \min_{x \in [(n-1)L_2\epsilon^\alpha, nL_2\epsilon^\alpha]} h\left(\frac{x}{\epsilon^\alpha}\right), \quad n = 1, 2 \ldots, N_\epsilon \tag{3.13}$$

and $\gamma_{n,\epsilon} \in [(n-1)L_2\epsilon^\alpha, nL_2\epsilon^\alpha]$ a point where the minimum (3.13) is attained, that is, $h(\frac{\gamma_{n,\epsilon}}{\epsilon^\alpha}) = h_{n,\epsilon}$ where $\gamma_{n,\epsilon}$ does not need to be uniquely defined. By extension, let us denote by $\gamma_{0,\epsilon} = 0$ and $\gamma_{N_\epsilon+1,\epsilon} = 1$.

Note that the set $\{\gamma_{0,\epsilon}, \gamma_{1,\epsilon}, ..., \gamma_{N_\epsilon+1,\epsilon}\}$ defines a partition for the unit interval $[0,1]$. Moreover, due to that $h(\cdot)$ is L_2-periodic we have that $h_{n,\epsilon} = h_0$ for $n = 1, 2 \ldots, N_\epsilon$.

We define now the test functions as follows. With $\phi \in H^1(0,1)$, we consider $\varphi^\epsilon \in H^1(\widetilde{\Omega}^\epsilon)$ defined as

$$\varphi^\epsilon(x_1, x_2) = \begin{cases} X_n^\epsilon(x_1, x_2), & (x_1, x_2) \in \widetilde{\Omega}_-^\epsilon \cap Q_n^\epsilon, \quad n = 1, 2, \ldots \\ \phi(x_1), & (x_1, x_2) \in \widetilde{\Omega}_+^\epsilon \equiv \Omega_0 \end{cases} \tag{3.14}$$

where Q_n^ϵ is the rectangle $Q_n^\epsilon = \{(x_1, x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon},\ -h_1 < x_2 < -h_0\}$ and the function X_n^ϵ is the solution of the problem

$$\begin{cases} -\dfrac{\partial^2 X_n^\epsilon}{\partial x_1^2} - \dfrac{1}{\epsilon^2} \dfrac{\partial^2 X_n^\epsilon}{\partial x_2^2} = 0, \quad \text{in } Q_n^\epsilon \\ \quad \dfrac{\partial X_n^\epsilon}{\partial N^\epsilon} = 0, \quad \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon \\ \quad X_n^\epsilon(x_1, x_2) = \phi(x_1), \quad \text{on } \Gamma_n^\epsilon \end{cases} \tag{3.15}$$

where Γ_n^ϵ is the base of the rectangle, that is, $\Gamma_n^\epsilon = \{(x_1, -h_0) : \gamma_{n,\epsilon} \le x_1 \le \gamma_{n+1,\epsilon}\}$.

From Lemma 3.1 we have

$$\left\| \frac{\partial X_n^\epsilon}{\partial x_1^2} \right\|_{L^2(Q_n^\epsilon)}^2 + \frac{1}{\epsilon^2} \left\| \frac{\partial X_n^\epsilon}{\partial x_2^2} \right\|_{L^2(Q_n^\epsilon)}^2 \le C\epsilon^{\alpha-1} \|\phi'\|_{L^2(\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon})}^2. \tag{3.16}$$

Furthermore, since

$$\varphi^\epsilon(x_1, x_2) - \phi(x_1) = \varphi^\epsilon(x_1, x_2) - \varphi^\epsilon(x_1, 0) = \int_0^{x_2} \frac{\partial \varphi^\epsilon}{\partial x_2}(x_1, s)\, ds,$$

we have by (3.14) and (3.16) that

$$\|\varphi^\epsilon - \phi\|_{L^2(\widetilde{\Omega}^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{3.17}$$

(e) Passing to the limit.

We can now pass to the limit in (3.3) by making use of test functions φ^ϵ defined above. For this, we study the convergence of each term in (3.3).

- First integrand:

$$\int_{\Omega_0} \left\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \varphi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 \to \int_{\Omega_0} \xi^*(x_1, x_2) \phi'(x_1)\, dx_1 dx_2 \text{ as } \epsilon \to 0. \tag{3.18}$$

  Thanks to the choice of the test function (3.14) and the convergence (3.11), we easily get (3.18).

- Second integrand:

$$\int_{\widetilde{\Omega}_-^\epsilon} \left\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \varphi^\epsilon}{\partial x_2} \right\} dx_1 dx_2 \to 0 \text{ as } \epsilon \to 0. \tag{3.19}$$

  From the definition of φ^ϵ, the Cauchy-Schwarz inequality and the inequality (3.16) we have (3.19).

- Third integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon\, \varphi^\epsilon\, dx_1 dx_2 \to \int_0^1 p\, u_0(x_1)\, \phi(x_1)\, dx_1 + \int_{\Omega_0} \theta(x_2)\, u_0(x_1)\, \phi(x_1)\, dx_1 dx_2 \text{ as } \epsilon \to 0 \tag{3.20}$$

  where the constant p is given by $p = \frac{1}{L_2} \int_0^{L_2} h(s) ds - h_0$.

  For this, note that we can rewrite the integral of the left side of (3.20) as

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon\, \varphi^\epsilon\, dx_1 dx_2 = \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon \left(P_\epsilon u^\epsilon - u_0\right) \varphi^\epsilon\, dx_1 dx_2 \\ + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \, \left(\varphi^\epsilon - \phi\right)\, dx_1 dx_2 + \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0\, \phi\, dx_1 dx_2.$$

From (3.10) and (3.17), we have that the first two terms in the right hand side above go to 0. Moreover, since

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon u_0 \,\phi \,dx_1 dx_2 = \int_{\widetilde{\Omega}_-^\epsilon} u_0 \,\phi \,dx_1 dx_2 + \int_{\Omega_0} \chi^\epsilon u_0 \,\phi \,dx_1 dx_2$$
$$= \int_0^1 u_0 \,\phi \left(h \left(\frac{x_1}{\epsilon^\alpha} \right) - h_0 \right) dx_1 + \int_{\Omega_0} \chi^\epsilon u_0 \,\phi \,dx_1 dx_2$$

we get (3.20) from the Average Theorem and (3.12) .

- Fourth integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon \,\varphi^\epsilon \,dx_1 dx_2 \to \int_0^1 \hat{f}(x_1) \,\phi(x_1) \,dx_1 \text{ as } \epsilon \to 0. \tag{3.21}$$

From (3.17) and the hypotheses of the theorem we have (3.21).

Therefore, using the convergences (3.19), (3.18), (3.20) and (3.21), we obtain the following limit variational formulation:

$$\int_{\Omega_0} \{\xi^*(x_1,x_2)\,\phi'(x_1) + \theta(x_2)\,u_0(x_1)\,\phi(x_1)\}\,dx_2 dx_1 + \int_0^1 p\,u_0(x_1)\,\phi(x_1)\,dx_1$$
$$= \int_0^1 \hat{f}(x_1)\,\phi(x_1)\,dx_1, \;\forall \phi \in H^1(0,1). \tag{3.22}$$

with $p = \frac{1}{L_2}\int_0^{L_2} h(s)ds - h_0$.

At this point the question is how to relate u_0 to ξ^*. In the following subsection we will show a equation for ξ^*.

(f) Relation between ξ^* and u_0.

Let us consider the following families of isomorphisms $T_k^\epsilon : A_k^\epsilon \mapsto Y$ given by

$$T_k^\epsilon(x_1,x_2) = (\frac{x_1 - \epsilon k L_1}{\epsilon}, x_2) \tag{3.23}$$

where

$$A_k^\epsilon = \{(x_1,x_2) \in \mathbb{R}^2 \mid \epsilon k L_1 \le x_1 < \epsilon L_1(k+1), -h_0 < x_2 < g_1\} \quad \text{and} \quad Y = (0,L_1) \times (-h_0, g_1).$$

with $k \in \mathbb{N}$. We can considerer extension operators $P \in \mathcal{L}(H^1(Y^*), H^1(Y)) \cap \mathcal{L}(L^2(Y^*), L^2(Y))$,the proof is done in [3]. Using these operators, the isomorphism (3.23) and the unique solution of the auxiliary problem (2.9) we define ω_k^ϵ in $(x_1,x_2) \in A_k^\epsilon$ by

$$\omega_k^\epsilon(x_1,x_2) \;=\; x_1 - \epsilon\Big(PX \circ T_k^\epsilon(x_1,x_2)\Big) = x_1 - \epsilon\Big(PX(\frac{x_1 - \epsilon L_1 k}{\epsilon}, x_2)\Big).$$

Observe that for any $(x_1,x_2) \in \widetilde{\Omega}_+^\epsilon$ there is k such that $(x_1,x_2) \in A_k^\epsilon$. Therefore, the function $\omega^\epsilon(x_1,x_2) = \omega_k^\epsilon(x_1,x_2)$ is well defined and $\omega^\epsilon \in H^1(\widetilde{\Omega}_+^\epsilon)$. We introduce now the vector $\eta^\epsilon = (\eta_1^\epsilon, \eta_2^\epsilon)$ defined by

$$\eta_i^\epsilon(x_1,x_2) = \frac{\partial \omega^\epsilon}{\partial x_i}(x_1,x_2), \quad (x_1,x_2) \in \Omega_+^\epsilon \tag{3.24}$$

where $\Omega_+^\epsilon = \{(x_1,x_2) \in \mathbb{R}^2 \;:\; 0 < x_1 < 1 \text{ and } -h_0 < x_2 < g(x_1/\epsilon)\}$.

Taking into account the definition of X if we consider a test function $\psi \in H^1(\Omega_+^\epsilon)$ with $\psi = 0$ in neighborhood of the lateral boundaries, we get

$$\int_{\Omega_+^\epsilon} \left(\eta_1^\epsilon \frac{\partial \psi}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2} \frac{\partial \psi}{\partial x_2} \right) dx_1 dx_2 = 0. \tag{3.25}$$

Then, with the variational formulation (3.1) and the identity (3.25) we can write:

$$\int_{\widetilde{\Omega}^\epsilon} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \varphi}{\partial x_2} + \chi^\epsilon P_\epsilon u^\epsilon \varphi \Big\} dx_1 dx_2 - \int_{\Omega_+^\epsilon} \left(\eta_1^\epsilon \frac{\partial \psi}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2} \frac{\partial \psi}{\partial x_2} \right) dx_1 dx_2$$

$$= \int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon f^\epsilon \varphi dx_1 dx_2, \ \forall \varphi \in H^1(\Omega^\epsilon). \tag{3.26}$$

We would like to pass to the limit in this expression. For this, we will construct appropriate test functions, which used in the identity (3.26) allow us to pass to the limit in all the terms.

(g) Limit of ω^ϵ and η_1^ϵ. From the definition of ω^ϵ, we have

$$\omega^\epsilon \to x_1, \ s - L^2(\Omega_0); \qquad \frac{\partial \omega^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega_0); \qquad \widetilde{\eta}_1^\epsilon \rightharpoonup q, \ w^* - L^\infty(\Omega_0), \tag{3.27}$$

where

$$q(x_2) := \frac{1}{L_1} \int_0^{L_1} \Big(1 - \widetilde{\frac{\partial X}{\partial y_1}}(s, x_2)\Big) \chi(s, x_2) ds$$

See [2] for more details.

(h) Function test.

Let $\phi = \phi(x_1) \in \mathcal{C}_0^\infty(0,1)$. We introduce the test function

$$\psi^\epsilon(x_1, x_2) = \begin{cases} X_n^\epsilon(x_1, x_2), & (x_1, x_2) \in \widetilde{\Omega}_-^\epsilon \cap Q_n^\epsilon, \quad n = 1, 2, \dots \\ \phi(x_1)\omega^\epsilon(x_1, x_2), & (x_1, x_2) \in \widetilde{\Omega}_+^\epsilon \equiv \Omega_0, \end{cases} \tag{3.28}$$

where ω_ϵ is defined above and, as in (3.14), Q_n^ϵ is the rectangle $Q_n^\epsilon = \{(x_1, x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon}, \ -h_1 < x_2 < -h_0\}$ and the function X_n^ϵ is the solution of the problem

$$\begin{cases} -\dfrac{\partial^2 X_n^\epsilon}{\partial x_1^2} - \dfrac{1}{\epsilon^2} \dfrac{\partial^2 X_n^\epsilon}{\partial x_2^2} = 0, \quad \text{in } Q_n^\epsilon \\ \dfrac{\partial X_n^\epsilon}{\partial N^\epsilon} = 0, \quad \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon \\ X_n^\epsilon(x_1, x_2) = \phi(x_1)\omega^\epsilon(x_1, -h_0), \quad \text{on } \Gamma_n^\epsilon \end{cases} \tag{3.29}$$

where Γ_n^ϵ is the base of the rectangle, that is, $\Gamma_n^\epsilon = \{(x_1, -h_0) : \gamma_{n,\epsilon} \le x_1 \le \gamma_{n+1,\epsilon}\}$.

Moreover, we define the function $X^\epsilon(x_1, x_2) = X_n^\epsilon(x_1, x_2)$ as $(x_1, x_2) \in Q_n^\epsilon \cap \overline{\Omega}_-^\epsilon$.

From Lemma 3.1 and using the the properties of ω^ϵ we have that the function X^ϵ is $H^1(\widetilde{\Omega}_-^\epsilon)$ and satisfies the following estimate

$$\left\| \frac{\partial X^\epsilon}{\partial x_1} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial X^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} \le C\, \epsilon^{\alpha - 1} \left\| \frac{\partial \big(\phi(x_1)\omega^\epsilon(x_1, -h_0)\big)}{\partial x_1} \right\|^2_{L^2(0,1)} \le \widetilde{C}\, \epsilon^{\alpha-1}. \tag{3.30}$$

where $\widetilde{C}$ denotes a constant independent of ϵ. Now, we can argue as in (3.17) and we obtain

$$\|\psi^\epsilon - \phi \overline{P} \omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{3.31}$$

where $\overline{P}\omega^\epsilon$ is the function defined on the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1), \ -h_1 < x_2 < g_1\}$ using a extension operator obtained by reflection in the negative vertical direction along the line $x_2 = -h_0$. .Indeed, since

$$\psi^\epsilon(x_1, x_2) - \phi(x_1)\overline{P}\omega^\epsilon(x_1, x_2) = \psi^\epsilon(x_1, x_2) - \phi(x_1)\omega^\epsilon(x_1, -x_2 - 2h_0)$$
$$= \psi^\epsilon(x_1, x_2) - \psi^\epsilon(x_1, -x_2 - 2h_0) = \int_{-x_2-2h_0}^{x_2} \tfrac{\partial \psi^\epsilon}{\partial x_2}(x_1, s) d_s \quad \text{for } (x_1, x_2) \in \widetilde{\Omega}_-^\epsilon$$

we have by (3.28)

$$\|\psi^\epsilon - \phi \overline{P} \omega^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)} \le C(g, h) \left\| \frac{\partial \psi^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}^\epsilon)} = C(g, h) \left\| \frac{\partial \omega^\epsilon}{\partial x_2} \phi \right\|^2_{L^2(\widetilde{\Omega}_+^\epsilon)} + C(g, h) \left\| \frac{\partial X^\epsilon}{\partial x_2} \right\|^2_{L^2(\widetilde{\Omega}_-^\epsilon)} \to 0 \text{ as } \epsilon \to 0.$$

(i) Passing to the limit.

Now we pass to the limit in the equality (3.26) considering the test functions $\varphi = \psi^\epsilon$ and $\psi = \phi u^\epsilon$.

- First integrand:

$$\int_{\widetilde{\Omega}^\epsilon_-} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \psi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \to 0 \text{ as } \epsilon \to 0. \tag{3.32}$$

Taking account the definition of ψ^ϵ, the Cauchy-Schwarz inequality and the estimate (3.30) we obtain the convergence (3.32).

- Second integrand:

$$\int_{\widetilde{\Omega}^\epsilon_+} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \psi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \frac{\partial \psi^\epsilon}{\partial x_2} - \eta_1^\epsilon \frac{\partial(\phi u^\epsilon)}{\partial x_1} - \eta_2^\epsilon \frac{1}{\epsilon^2} \frac{\partial(\phi u^\epsilon)}{\partial x_2} \Big\} dx_1 dx_2$$
$$\to \int_{\Omega_0} \Big\{ \xi^* \frac{\partial \phi}{\partial x_1} x_1 - q \frac{\partial \phi}{\partial x_1} u_0 \Big\} dx_1 dx_2 \text{ as } \epsilon \to 0. \tag{3.33}$$

From the definitions of η_i^ϵ and ψ^ϵ the second integrand reduces to $\int_{\widetilde{\Omega}^\epsilon_+} \Big\{ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \frac{\partial \phi}{\partial x_1} \omega^\epsilon - \widetilde{\eta}_1^\epsilon \frac{\partial \phi}{\partial x_1} P_\epsilon u^\epsilon \Big\} dx_1 dx_2$.

Therefore, using convergences (3.10), (3.11) and (3.27), we have (3.33).

- Third integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \chi^\epsilon P_\epsilon u^\epsilon \, \psi^\epsilon \, dx_1 dx_2 \to \int_0^1 p \, u_0(x_1) \, \phi(x_1) \, x_1 \, dx_1 + \int_{\Omega_0} \theta(x_2) \, u_0(x_1) \, \phi(x_1) \, x_1 \, dx_1 dx_2, \text{ as } \epsilon \to 0. \tag{3.34}$$

Following along the lines of the proof of the convergence (3.20) we have this convergence.

- Fourth integrand:

$$\int_{\widetilde{\Omega}^\epsilon} \widetilde{f}^\epsilon \, \psi^\epsilon \, dx_1 dx_2 \to \int_0^1 \hat{f}(x_1) \, \phi(x_1) \, x_1 dx_1 \text{ as } \epsilon \to 0. \tag{3.35}$$

Using the same computations as those made to derive (3.21) we obtain (3.35)

Now, by the convergences shown in (3.32), (3.33), (3.34) and (3.35), we can pass to the limit in (3.26) considering the test functions $\varphi = \psi^\epsilon$ and $\psi = \phi u^\epsilon$. More precisely, we have

$$\int_{\Omega_0} \Big\{ \xi^* \frac{\partial \phi}{\partial x_1} x_1 - q \frac{\partial \phi}{\partial x_1} u_0 \Big\} dx_1 dx_2 + \int_0^1 p \, u_0 \, \phi \, x_1 \, dx_1 + \int_{\Omega_0} \theta \, u_0 \, \phi \, x_1 \, dx_1 dx_2 = \int_0^1 \hat{f} \, \phi \, x_1 \, dx_1 \quad \forall \phi \in \mathcal{C}_0^\infty(0,1) \tag{3.36}$$

where p and q are given by

$$p = \frac{1}{L_2} \int_0^{L_2} h(s) ds - h_0, \quad q(x_2) = \frac{1}{L_1} \int_0^{L_1} \Big(1 - \widetilde{\frac{\partial X}{\partial y_1}}(s, x_2) \Big) \chi(s, x_2) ds.$$

Taking the test function ϕx_1 in (3.22) we obtain

$$\int_{\Omega_0} \xi^*(x_1, x_2) \frac{\partial}{\partial x_1}(\phi x_1) dx_2 dx_1 + \int_0^1 p \, u_0(x_1) \, \phi(x_1) \, x_1 \, dx_1 + \int_{\Omega_0} \theta(x_2) \, u_0(x_1) \, \phi(x_1) \, x_1 \, dx_1 dx_2$$
$$= \int_0^1 \hat{f}(x_1) \, \phi(x_1) \, x_1 \, dx_1 \tag{3.37}$$

Due to $\xi^* \frac{\partial}{\partial x_1}(\phi \, x_1) = \xi^* x_1 \frac{\partial \phi}{\partial x_1} + \xi^* \phi$, we can rewrite (3.36) as

$$\int_{\Omega_0} \Big\{ \xi^* \frac{\partial}{\partial x_1}(\phi x_1) - \phi \xi^* - q \frac{\partial \phi}{\partial x_1} u_0 \Big\} dx_1 dx_2 + \int_0^1 p \, u_0 \, \phi \, x_1 \, dx_1 + \int_{\Omega_0} \theta \, u_0 \, \phi \, x_1 \, dx_1 dx_2$$
$$= \int_0^1 \hat{f} \, \phi \, x_1 \, dx_1 \; \forall \phi \in \mathcal{C}_0^\infty(0,1). \tag{3.38}$$

Therefore, it follows from (3.37) and (3.38) that, for all $\phi \in \mathcal{C}_0^\infty(0,1)$

$$0 = \int_{\Omega_0} \Big\{ \phi \xi^* + q \frac{\partial \phi}{\partial x_1} u_0 \Big\} dx_1 dx_2 = \int_{\Omega_0} \Big\{ \phi \xi^* - q \frac{\partial u_0}{\partial x_1} \phi \Big\} dx_1 dx_2 \tag{3.39}$$

With the definition of $\hat{q}$ given by (2.8) and performing an iterated integration in (3.39) we obtain

$$\int_0^1 \phi(x_1)\Big(\int_{-h_0}^{g_1} \xi^*(x_1,x_2)dx_2 - \hat{q}\frac{\partial u_0}{\partial x_1}\Big)dx_1 = 0 \qquad \forall\, \phi \,\in\, \mathcal{C}_0^\infty(0,1)$$

So, the equation satisfied by ξ^* is:

$$\int_{-h_0}^{g_1} \xi^*(x_1,x_2)dx_2 = \hat{q}\frac{\partial u_0}{\partial x_1}$$

The last step is placing this last equality in (3.22). We get

$$\int_0^1 \Big\{\hat{q}\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L_1}\, u_0\,\varphi + u_0\,\varphi\, p\Big\}dx_1 = \int_0^1 \hat{f}\,\varphi\, dx_1, \quad \forall \varphi \in H^1(0,1). \tag{3.40}$$

Hence u_0 is the unique solution of (3.40), and we obtain that any convergent subsequence of $\{u^\epsilon\}$ tends to this unique solution. This complete the proof of Theorem 2.2. □

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